STUDY SET 2

Continuous Probability Distributions

1. The normal distribution is used to approximate the binomial under certain conditions. What is the best way to approximate the binomial using the normal?

ANSWER:
When the number of trials n is large enough such that $np(1-p) > 5$, where p is the probability of success in any trial, the binomial distribution often looks like a bell-shaped curve and it is approximated by a normal distribution with the same mean and variance. In this case, the binomial probabilities can be approximated by the probabilities under the normal curve. The best way to approximate a discrete random variable, such as the binomial, by a continuous random variable, such as the normal. The normal probabilities are calculated as $Z = \frac{X - nP}{\sqrt{nP(1-P)}}$.

2. Suppose that 24% of all sales in a grocery store are for amounts greater than $100. In a random sample of 50 invoices, what is the probability that more than ten of the invoices are for over $100? Use the normal approximation for the binomial distribution.

ANSWER:
Without continuity correction $P(X>10) = P(Z>-0.66) = 0.7454$.

3. Investment *A* has an expected return of 10% with a standard deviation of 3.5%. Investment *B* has an expected return of 6% with a standard deviation of 1.2%. If you invest equally in both investments, what is the expected return and standard deviation of your portfolio? What assumptions have you made?

ANSWER:
$E(A+B) = 0.10 + 0.06 = 0.16$
Assume the rates of return are independent, $\text{Var}(A+B) = \text{Var}(A) + \text{Var}(B) = 13.69$. Hence, σ of $(A+B) = 3.7$

<u>THE NEXT THREE QUESTIONS ARE BASED ON THE FOLLOWING INFORMATION:</u>
The amount of time you have to wait at a dentist's office before you are called in is uniformly distributed between zero and twenty minutes.

4. What is the probability that you have to wait more than 8 minutes?

ANSWER: 0.60

5. What is the probability that you have to wait between 10 and 15 minutes?

ANSWER: 0.25

6. Seventy percent of the time, you will be called in before you have to wait how long?

ANSWER:
The range is a total of 20 minutes. 70% of the time, you should be called in before you wait 14 minutes.

THE NEXT TWO QUESTIONS ARE BASED ON THE FOLLOWING INFORMATION:
The average amount of time between a score by either team in college soccer games is 15.2 minutes. Suppose that the time between scores follows an exponential distribution.

7. What is the probability that a soccer game goes for 45 minutes without either team scoring?

ANSWER: 0.0518

8. The length of time it takes to fill an order at a local sandwich shop is normally distributed with a mean of 4.1 minutes and a standard deviation of 1.3 minutes. If the sandwich shop employees make $6.00 an hour, what is the mean and standard deviation for the labor costs per sandwich?

ANSWER: μ_X = 4.1 and σ_X^2 = 1.69. Labor cost per minute = $0.10, $Y = 0.10X$.

$\mu_Y = 0.10\,\mu_X = (0.10)(4.1) = \$0.41$, $\sigma_Y^2 = (0.10)^2\,\sigma_X^2 = 0.0169$. Hence, $\sigma_Y = 0.13$.

9. You are the owner of a small casino in Las Vegas. You want to reward the high-rollers who come to your casino. You want to give free accommodations to no more than 10% of your patrons. Suppose that the mean amount wagered by all patrons is $287, with a standard deviation of $15. You should give free accommodations to those individuals who wager over how much money?

ANSWER: $306.20

10. You are in charge of arranging the catering for a company meeting. Your company is responsible for paying for all meals ordered, so you don't want to order too many. Suppose that the expected number of people for the meeting is 84, with a standard deviation of 4 people. What is the fewest number of meals should you order so that the probability of having more people than meals is 5%?

ANSWER: ≈ 91 meals

11. It has been found that 62.1% of all unsolicited third class mail delivered to households goes unread. If, over the course of a month, a household receives 150 pieces of unsolicited mail, what is the probability that the household discards more than 80 pieces of the mail without reading it? Use the normal approximation for the binomial distribution.

ANSWER: 0.9864

THE NEXT FOUR QUESTIONS ARE BASED ON THE FOLLOWING INFORMATION:
Let the random variable Z follow a standard normal distribution.

12. What is $P(Z > 0.29)$?

ANSWER:
0.38597

13. What is $P(Z < 1.23)$?

ANSWER:
0.8907

14. What is $P(Z > -0.52)$?

ANSWER:
0.6985

15. What is $P(-0.44 < Z < 1.2)$?

ANSWER:
0.5549

16. As manager of a pizza shop, you are responsible for placing the food orders. You currently have enough anchovies for 8 pizzas. You expect to have orders for 60 pizzas tonight. If 8% of all pizzas are ordered with anchovies, what is the probability that you run out of anchovies before the evening is over? Use the normal approximation for the binomial.

ANSWER: 0.0643

THE NEXT FOUR QUESTIONS ARE BASED ON THE FOLLOWING INFORMATION:
Let the random variable Z follow a standard normal distribution.

17. Find the value k such that $P(Z > k) = 0.83$.

ANSWER: $k = -0.95$

18. Find the value k such that $P(Z > k) = 0.43$.

ANSWER: $k = 0.18$

19. Find the value k such that $P(0 < Z < k) = 0.35$.

ANSWER:
$k = 1.04$

20. Find the value k such that $P(-0.71 < Z < k) \approx 0.67$.

ANSWER:
$k = 1.33$

21. The time it takes to assemble an electronic component is normally distributed with a mean of 17.2 minutes and a standard deviation of 3.1 minutes. The probability is 90% that it will take at least how long to assemble a component?

ANSWER: 13.23 minutes

THE NEXT TWO QUESTIONS ARE BASED ON THE FOLLOWING INFORMATION:
The stamping machine on a production line periodically is taken off-line for maintenance. Assume that the amount of time the machine is off-line is uniformly distributed between 15 and 30 minutes.

22. What is the probability that the machine is off-line for more than 18 minutes?

ANSWER: 0.80.

23. What is the probability that the machine is off-line between 21 and 27 minutes?

ANSWER: 0.40

24. In a recent survey of high school students, it was found that the average amount of money spent on entertainment each week was normally distributed with a mean of $52.30. Suppose you are told that there is an 80% probability that a randomly-selected student spends somewhere between $49.74 and $54.86. What is the standard deviation of the amount of money spent by high school students monthly?

ANSWER: 2.0

THE NEXT TWO QUESTIONS ARE BASED ON THE FOLLOWING INFORMATION:
A wire-spinning machine will spin, on average, 12.3 miles before needing maintenance. Assume the time between maintenance is exponentially distributed.

25. What is the probability that a spinning machine just placed back in service will need maintenance before it produces 4 miles of wire?

ANSWER: 0.2776

26. What is the median amount of time before the machine needs servicing?

ANSWER: 8.53 miles

27. You are the Webmaster for your firm's Website. From your records, you know that the probability that a visitor will buy something from your firm is 0.23. If the number of visitors in one day is 952, what is the probability that less than 200 of them will buy something from your firm? Use the normal approximation for the binomial.

ANSWER: 0.0721

THE NEXT TWO QUESTIONS ARE BASED ON THE FOLLOWING INFORMATION:
The life of a new type of light bulb is uniformly distributed between 1,200 and 1,600 hours.

28. The probability is 70% that a randomly-selected light bulb will last at least how long.

ANSWER: 1480

29. What is the probability that a randomly-selected light bulb burns out in less than 1,500 hours?

ANSWER: 0.75

30. Sales at a local plumbing wholesaler consist of both over-the-counter sales as well as deliveries. During the course of a month, over-the-counter sales have a mean of \$102,972 with a standard deviation of \$13,523. Over the same time period, deliveries average \$242,354 with a standard deviation of \$24,956. Assuming that the sales over-the-counter are independent of deliveries, what are the mean and standard deviation of the wholesaler monthly sales?

ANSWER:
Let X = over-the-counter sales, Let Y = deliveries, and W=Wholesaler's monthly sales
$E(W) = E(X + Y) = E(X) + E(Y) = 102972 + 242354 = \$345,326$
$\sigma_W^2 = \text{Var}(W) = \text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) = 805,673,465$, $\sigma_W = \$28,384$

31. During hours, students arrive, on average, every ten minutes. Assume that the distribution of the time between arrivals follows an exponential distribution. Suppose that a customer has just left the gas station. What is the probability that the cashier at the gas station has more than 15 minutes before the next customer arrive?

ANSWER:
Exponential with mean $1/\lambda$ =10, then, λ = 0.10, and $P(T>15) = e^{-15\lambda} = e^{-1.5} = 0.223$

32. The number of viewers ordering a particular pay-per-view program is normally distributed. 20% of the time, fewer than 20,000 people order the program. Only ten percent of the time more than 28,000 people order the program. What is the mean and standard deviation of the number of people ordering the program?

ANSWER:
Mean = 23,170, standard deviation = 3,774.

THE NEXT FIVE QUESTIONS ARE BASED ON THE FOLLOWING INFORMATION:
Let the random variable X follow a normal distribution with mean μ = 48 and variance σ^2 = 60.84.

33. Find the probability that X is greater than 58.

ANSWER: 0.1003

34. Find the probability that X is greater than 36 and less than 60.

ANSWER:
0.8764

35. Find the probability that X is less than 52.

ANSWER: 0.695

36. The probability is 0.2 that X is greater than what number?

ANSWER: 54.552

37. The probability is approximately 0.05 that X is in the symmetric interval about the mean between which two numbers?

ANSWER:
$P(a < X < b) = 0.05 \Rightarrow P[(a - 48) / 7.8 < Z < (b - 48) / 7.8] = 0.05$
$\Rightarrow (a - 48) / 7.8 = -0.06$ and $(b - 48) / 7.8 = 0.06$
$\Rightarrow$ a = 47.532 and b = 48.468.

38. The binomial distribution and the normal distribution are similar in a number of ways, although they are different distributions. What are the basic differences between the two distributions?

ANSWER:
The binomial distribution is a discrete probability distribution. Outcomes are integers 1, 2, 3,…n where n is the number of trials for the binomial random variable. The normal distribution is a continuous probability distribution with a bell-shaped curve describing the shape of the distribution. A normal random variable can take on an infinite number of values within an interval.

THE NEXT THREE QUESTIONS ARE BASED ON THE FOLLOWING INFORMATION:
A professor sees students during regular office hours. Times spent with students follow an exponential distribution with mean of 12 minutes.

39. Find the probability that a given student spends less than 15 minutes with the professor.

ANSWER:
$P(X < 15) = 1 - e^{-(1/12)15} = 1 - e^{-1.25} = 0.7135$

40. Find the probability that a given student spends more than 8 minutes with the professor.

ANSWER:
$P(X > 8) = 1 - [1 - e^{-(1/12)8}] = e^{-0.667} = 0.5132$

41. Find the probability that a given student spends between 12 and 15 minutes with the professor.

ANSWER: 0.0814

42. The total cost for a production process is equal to $1,200 plus 2.5 times the number of units produced. The mean and variance for the number of units produced are 520 and 840, respectively. Find the mean and variance of the total cost.

ANSWER: 2500 and 5250.

43. Given the random variables *X* and *Y* have a correlation coefficient equal to 0.50, find the mean and variance of the random variable W = 4X+3Y.

ANSWER: 500 and 3,799.977

44. Given the random variables *X* and *Y* have a correlation coefficient equal to 0.50, find the mean and variance of the random variable W = 4X - 3Y.

ANSWER: -100 and 1,400.023

45. Given the random variables *X* and *Y* have a correlation coefficient equal to -0.50, find the mean and variance of the random variable W = 4X+3Y.

ANSWER: 500 and 1,400.023

46. Given the random variables *X* and *Y* have a correlation coefficient equal to -0.50, find the mean and variance of the random variable W = 4X - 3Y.

ANSWER: -100 and 3,799.977

47. The profit for a production process is equal to $7,200 minus 2.8 times the number of units produced. The mean and variance for the number of units produced are 1,100 and 800, respectively. Find the mean and variance of the profit.

ANSWER: 6272

THE NEXT TWELVE QUESTIONS ARE BASED ON THE FOLLOWING INFORMATION:
Consider a random sample of size n =1800 from a binomial probability distribution with P = 0.40, and X = number of successes.

48. Find the mean and standard deviation of the number of successes.

ANSWER:
Mean = $E(X) = \mu = nP =$ (1800)(0.40) = 720
Standard deviation = $\sigma = \sqrt{nP(1-P)} = \sqrt{(1800)(0.40)(0.60)} = 20.785$

49. Find the probability that the number of successes is greater than 775.

ANSWER: 0.004

50. Find the probability that the number of successes is greater than 775.

ANSWER: 0.004

51. Find the probability that the number of successes is less than 700.

ANSWER: 0.1685

52. Find the probability that the number of successes is between 680 and 750.

ANSWER: 0.8977

53. With probability 0.09 the number of successes is less than how many?

ANSWER: $\Rightarrow$a = 692.02 □ 692

54. With probability 0.20, the number of successes is greater than how many?

ANSWER: $\Rightarrow$a = 737.54 or 734

55. Find the mean and standard deviation of the proportion of successes.

ANSWER:
Mean = $\mu = P$ = 0.40,
Standard deviation = $\sigma = \sqrt{P(1-P)/n} = \sqrt{(0.40)(0.60)/1800} = 0.01155$

56. Find the probability that the percentage of successes is greater than 0.42.

ANSWER: 0.0418

57. Find the probability that the percentage of successes is less than 0.37.

ANSWER: 0.0047

58. Find the probability that the percentage of successes is between 0.38 and 0.43.

ANSWER: 0.9535

59. With probability 0.20, the percentage of successes is less than what value?

ANSWER:
$P(P < \text{a} \mid n = 1800, P = 0.40) \approx P(P < \text{a} \mid \mu = 0.40, \sigma = 0.01155) = 0.20$

$\Rightarrow P[Z < (\text{a} - 0.40) / 0.01155] = 0.20$

$\Rightarrow (\text{a} - 0.40) / 0.01155 = -0.84$

$\Rightarrow$a = 0.39 or 39%

60. With probability 0.09 the percentage of successes is greater than what value?

ANSWER:

$P(P > \text{a} \mid n = 1800, P = 0.40) \approx P(P > \text{a} \mid \mu = 0.40, \sigma = 0.01155) = 0.09$

$\Rightarrow P[Z > (\text{a} - 0.40) / 0.01155] = 0.09$

$\Rightarrow (\text{a} - 0.40) / 0.01155 = 1.34$

$\Rightarrow \text{a} = 0.415$ or 41.5%

61. A homeowner has installed a new energy-efficient furnace. It is estimated that over a year the new furnace will reduce energy costs by an amount that can be regarded as a random variable with a mean of \$265 and standard deviation of \$55. Stating any assumptions you need to make, find the mean and standard deviation of the total energy costs reductions over a period of 5 years.

ANSWER: 1325 and 275

THE NEXT THREE QUESTIONS ARE BASED ON THE FOLLOWING INFORMATION:

Suppose that the time between successive occurrences of an event follows an exponential distribution with mean $1/\lambda$ minutes. Assume that an event occurs.

62. Show that the probability that more than 4 minutes elapses before the occurrence of the next event is $e^{-4\lambda}$.

ANSWER: $e^{-4\lambda}$

63. Show that the probability that more than 8 minutes elapses before the occurrence of the next event is $e^{-8\lambda}$.

ANSWER: $e^{-8\lambda}$

64. Using the results of the previous two questions, show that if 4 minutes have already elapsed, the probability that a further 4 minutes will elapse before the next occurrence is $e^{-8\lambda}$. Explain your answer in words.

ANSWER: $e^{-4\lambda}$

THE NEXT FOUR QUESTIONS ARE BASED ON THE FOLLOWING INFORMATION:

A continuous random variable X has the probability density function: $f(x) = 2e^{-2x}, \ x \geq 0$.

65. What is the distribution of the random variable X?

ANSWER: Exponential distribution with $\lambda = 2$

66. Find the mean and standard deviation of X.

ANSWER: Mean = Standard deviation = $1/\lambda = 0.5$

67. What is the probability that *x* is between 1 and 3?

ANSWER: 0.1329

68. What is the probability that *x* is at most 2?

ANSWER: 0.9817

69. When we convert normal distributions to the standard normal distribution, we are essentially making them all alike. How is this possible and what are the implications?

ANSWER:
Normal distributions are bell-shaped and symmetric around their means. When a normal population is standardized, every value is replaced by a value equal to the number of standard deviations above or below the mean that particular value is. So in a normal population with mean 10 and standard deviation 2, the value 8 becomes z= -1 and so does the value 6 in a normal population with mean 10 and standard deviation 4. So, all normal populations become related to this one standard normal population. That is why, for every normal population 68.26% fall within ± 1 standard deviation of the mean, 95.44% fall within 2 and 99.73% fall within ± 3. The only thing that really matters for a normal distribution are the standardized z values. Once you know how many standard deviations above or below the mean a value in the normal population is, you can always determine that value and any probabilities about that value

THE NEXT THREE QUESTIONS ARE BASED ON THE FOLLOWING INFORMATION:
The distribution of IQ scores for high school graduates is normally distributed with mean μ = 104 and standard deviation σ = 16.

70. Find the probability a person chosen at random from this group has an IQ score above 146.00.

ANSWER: 0.0043

71. What fraction of the IQ scores would be between 97 and 126?

ANSWER: 0.5862

72. What is the 95th percentile of this normal distribution?

ANSWER: 130.32.

THE NEXT FOUR QUESTIONS ARE BASED ON THE FOLLOWING INFORMATION:
Suppose x is normally distributed with a mean of 75 and a standard deviation of 4.

73. Find the 90th percentile.

ANSWER: 80.12

74. Find the 95th percentile.

ANSWER: 81.58

75. Find the 5th percentile.

ANSWER: 68.42

76. A normal random variable *x* has an unknown mean μ and standard deviation $\sigma = 2.5$. If the probability that *x* exceeds 7.5 is 0.8289, find μ.

ANSWER: 9.875.