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Applied Statistics and Probability for Engineers

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Chapter 8

Statistical Intervals for a Single Sample

8 Statistical Intervals for a Single Sample

CHAPTER OUTLINE

Learning Objectives for Chapter 8

After careful study of this chapter, you should be able to do the following:

1. Construct confidence intervals on the mean of a normal distribution, using normal distribution or *t* distribution method.
2. Construct confidence intervals on variance and standard deviation of normal distribution.
3. Construct confidence intervals on a population proportion.
4. Constructing an approximate confidence interval on a parameter.
5. Prediction intervals for a future observation.
6. Tolerance interval for a normal population.

8-1.1 Confidence Interval and its Properties

A **confidence interval** estimate for μ is an interval of the form

$$l \leq \mu \leq u,$$

where the end-points l and u are computed from the sample data.

There is a probability of $1 - \alpha$ of selecting a sample for which the CI will contain the true value of μ.

The endpoints or bounds l and u are called lower- and upper-confidence limits ,and $1 - \alpha$ is called the confidence coefficient.

Confidence Interval on the Mean, Variance Known

If $\overline{X}$ is the sample mean of a random sample of size *n* from a normal population with known variance σ^2, a $100(1 - \alpha)\%$ CI on μ is given by

$$\bar{x} - z_{\alpha/2}\sigma/\sqrt{n} \leq \mu \leq \bar{x} + z_{\alpha/2}\sigma/\sqrt{n} \qquad (8\text{-}1)$$

where $z_{\alpha/2}$ is the upper $100\alpha/2$ percentage point of the standard normal distribution.

EXAMPLE 8-1 Metallic Material Transition

Ten measurements of impact energy (*J*) on specimens of A238 steel cut at 60°C are as follows: 64.1, 64.7, 64.5, 64.6, 64.5, 64.3, 64.6, 64.8, 64.2, and 64.3. The impact energy is normally distributed with $\sigma = 1J$. Find a 95% CI for μ, the mean impact energy.

The required quantities are $z_{\alpha/2} = z_{0.025} = 1.96$, $n = 10$, $\sigma = 1$, and $\bar{x} = 64.46$.

The resulting 95% CI is found from Equation 8-1 as follows:

$$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

$$64.46 - 1.96\frac{1}{\sqrt{10}} \leq \mu \leq 64.46 + 1.96\frac{1}{\sqrt{10}}$$

$$63.84 \leq \mu \leq 65.08$$

Interpretation: Based on the sample data, a range of highly plausible values for mean impact energy for A238 steel at 60°C is

$$63.84J \leq \mu \leq 65.08J$$

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8.1.2 Sample Size for Specified Error on the Mean, Variance Known

If $\bar{x}$ is used as an estimate of μ, we can be 100(1 − α)% confident that the error $|\bar{x} - \mu|$ will not exceed a specified amount *E* when the sample size is

$$n = \left(\frac{z_{\alpha/2}\sigma}{E} \right)^2 \qquad (8\text{-}2)$$

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EXAMPLE 8-2 Metallic Material Transition

Consider the CVN test described in Example 8-1. Determine how many specimens must be tested to ensure that the 95% CI on μ for A238 steel cut at 60°C has a length of at most 1.0*J*.

The bound on error in estimation *E* is one-half of the length of the CI.
*U*se Equation 8-2 to determine *n* with $E = 0.5$, $\sigma = 1$, and $z_{\alpha/2} = 1.96$.

$$n = \left(\frac{z_{\alpha/2}\sigma}{E} \right)^2 = \left[\frac{(1.96)1}{0.5} \right]^2 = 15.37$$

Since, *n* must be an integer, the required sample size is $n = 16$.

8-1.3 One-Sided Confidence Bounds

A 100(1 − α)% **upper-confidence bound** for μ is

$$\mu \le \bar{x} + z_{\alpha}\sigma/\sqrt{n} \qquad (8\text{-}3)$$

and a 100(1 − α)% **lower-confidence bound** for μ is

$$\bar{x} - z_{\alpha}\sigma/\sqrt{n} = l \le \mu \qquad (8\text{-}4)$$

Example 8-3 One-Sided Confidence Bound

The same data for impact testing from Example 8-1 are used to construct a lower, one-sided 95% confidence interval for the mean impact energy.
Recall that z_α = 1.64, n = 10, σ = l, and $\bar{x} = 64.46$.

A 100(1 − α)% **lower-confidence bound** for μ is

$$\bar{x} - z_\alpha \frac{\sigma}{\sqrt{n}} \leq \mu$$

$$64.46 - 1.64 \frac{1}{\sqrt{10}} \leq \mu$$

$$63.94 \leq \mu$$

8-1.4 A Large-Sample Confidence Interval for μ

When *n* is large, the quantity

$$\frac{\overline{X} - \mu}{S/\sqrt{n}}$$

has an approximate standard normal distribution. Consequently,

$$\overline{x} - z_{\alpha/2}\frac{s}{\sqrt{n}} \leq \mu \leq \overline{x} + z_{\alpha/2}\frac{s}{\sqrt{n}} \qquad (8\text{-}5)$$

is a **large sample confidence interval** for μ, with confidence level of approximately $100(1 - \alpha)\%$.

Example 8-5 Mercury Contamination

A sample of fish was selected from 53 Florida lakes, and mercury concentration in the muscle tissue was measured (ppm). The mercury concentration values were

1.230	1.330	0.040	0.044	1.200	0.270
0.490	0.190	0.830	0.810	0.710	0.500
0.490	1.160	0.050	0.150	0.190	0.770
1.080	0.980	0.630	0.560	0.410	0.730
0.590	0.340	0.340	0.840	0.500	0.340
0.280	0.340	0.750	0.870	0.560	0.170
0.180	0.190	0.040	0.490	1.100	0.160
0.100	0.210	0.860	0.520	0.650	0.270
0.940	0.400	0.430	0.250	0.270	

Find an approximate 95% CI on μ.

Example 8-5 Mercury Contamination (continued)

The summary statistics for the data are as follows:

Variable	*N*	Mean	Median	StDev	Minimum	Maximum	*Q*1	*Q*3
Concentration	53	0.5250	0.4900	0.3486	0.0400	1.3300	0.2300	0.7900

Because $n > 40$, the assumption of normality is not necessary to use in Equation 8-5. The required values are $n = 53$, $\bar{x} = 0.5250$, $s = 0.3486$, and $z_{0.025} = 1.96$.

The approximate 95% CI on μ is

$$\bar{x} - z_{0.025}\frac{s}{\sqrt{n}} \le \mu \le \bar{x} + z_{0.025}\frac{s}{\sqrt{n}}$$

$$0.5250 - 1.96\frac{0.3486}{\sqrt{53}} \le \mu \le 0.5250 + 1.96\frac{0.3486}{\sqrt{53}}$$

$$0.4311 \le \mu \le 0.6189$$

<u>Interpretation:</u> This interval is fairly wide because there is variability in the mercury concentration measurements. A larger sample size would have produced a shorter interval.

Large-Sample Approximate Confidence Interval

Suppose that θ is a parameter of a probability distribution, and let $\hat{\theta}$ be an estimator of θ. Then a large-sample approximate CI for θ is given by

$$\hat{\theta} - z_{\alpha/2}\sigma_{\hat{\Theta}} \le \theta \le \hat{\theta} + z_{\alpha/2}\sigma_{\hat{\Theta}}$$

8-2.1 The *t* distribution

Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal distribution with unknown mean μ and unknown variance σ^2. The random variable

$$T = \frac{\overline{X} - \mu}{S/\sqrt{n}} \qquad (8\text{-}6)$$

has a *t* distribution with $n - 1$ degrees of freedom.

8-2.2 The Confidence Interval on Mean, Variance Unknown

If $\overline{x}$ and s are the mean and standard deviation of a random sample from a normal distribution with unknown variance σ^2, a **100(1 – α)% confidence interval on μ** is given by

$$\overline{x} - t_{\alpha/2,n-1} s/\sqrt{n} \le \mu \le \overline{x} + t_{\alpha/2,n-1} s/\sqrt{n} \quad (8\text{-}7)$$

where $t_{\alpha/2,n-1}$ the upper 100α/2 percentage point of the t distribution with n – 1 degrees of freedom.

One-sided confidence bounds on the mean are found by replacing $t_{\alpha/2,\text{n-1}}$ in Equation 8-7 with $t_{\alpha,\text{n-1}}$.

Example 8-6 Alloy Adhesion

Construct a 95% CI on μ to the following data.

19.8	10.1	14.9	7.5	15.4	15.4
15.4	18.5	7.9	12.7	11.9	11.4
11.4	14.1	17.6	16.7	15.8	
19.5	8.8	13.6	11.9	11.4	

The sample mean is $\bar{x} = 13.71$ and sample standard deviation is s = 3.55.

Since n = 22, we have $n - 1$ =21 degrees of freedom for t, so $t_{0.025,21}$ = 2.080.

The resulting CI is

$$\bar{x} - t_{\alpha/2,n-1}s/\sqrt{n} \le \mu \le \bar{x} + t_{\alpha/2,n-1}s/\sqrt{n}$$

$$13.71 - 2.080(3.55)/\sqrt{22} \le \mu \le 13.71 + 2.080(3.55)/\sqrt{22}$$

$$13.71 - 1.57 \le \mu \le 13.71 + 1.57$$

$$12.14 \le \mu \le 15.28$$

Interpretation: The CI is fairly wide because there is a lot of variability in the measurements. A larger sample size would have led to a shorter interval.

χ^2 Distribution

Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal distribution with mean μ and variance σ^2, and let S^2 be the sample variance. Then the random variable

$$X^2 = \frac{(n-1)S^2}{\sigma^2} \qquad (8\text{-}8)$$

has a chi-square (χ^2) distribution with $n - 1$ degrees of freedom.

Confidence Interval on the Variance and Standard Deviation

If s^2 is the sample variance from a random sample of n observations from a normal distribution with unknown variance σ^2, then **a 100(1 – α)% confidence interval on σ^2 is**

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}} \le \sigma^2 \le \frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}} \qquad (8\text{-}9)$$

where $\chi^2_{\alpha/2,n-1}$ and $\chi^2_{1-\alpha/2,n-1}$ are the upper and lower 100α/2 percentage points of the chi-square distribution with n – 1 degrees of freedom, respectively.

A **confidence interval for σ** has lower and upper limits that are the square roots of the corresponding limits in Equation 8–9.

One-Sided Confidence Bounds

The 100(1 – α)% lower and upper confidence bounds on σ^2 are

$$\frac{(n-1)s^2}{\chi^2_{\alpha,n-1}} \le \sigma^2 \text{ and } \quad \sigma^2 \le \frac{(n-1)s^2}{\chi^2_{1-\alpha,n-1}} \qquad (8\text{-}10)$$

Example 8-7 Detergent Filling

An automatic filling machine is used to fill bottles with liquid detergent. A random sample of 20 bottles results in a sample variance of fill volume of $s^2 = 0.0153^2$. Assume that the fill volume is approximately normal. Compute a 95% upper confidence bound.

A 95% upper confidence bound is found from Equation 8-10 as follows:

$$\sigma^2 \le \frac{(n-1)s^2}{\chi^2_{0.95,19}}$$

$$\sigma^2 \le \frac{(20-1)0.0153}{10.117}$$

$$\sigma^2 \le 0.0287$$

A confidence interval on the standard deviation σ can be obtained by taking the square root on both sides, resulting in

$$\sigma \le 0.17$$

Normal Approximation for Binomial Proportion

If *n* is large, the distribution of

$$Z = \frac{X - np}{\sqrt{np(1-p)}} = \frac{\hat{P} - p}{\sqrt{\frac{p(1-p)}{n}}}$$

is approximately standard normal.

The quantity $\sqrt{p(1-p)/n}$ is called the standard error of the point estimator $\hat{P}$.

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Approximate Confidence Interval on a Binomial Proportion

If $\hat{p}$ is the proportion of observations in a random sample of size *n*, an approximate 100(1 – α)**%** confidence interval on the proportion *p* of the population is

$$\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \qquad (8\text{-}11)$$

where $z_{\alpha/2}$ is the upper α/2 percentage point of the standard normal distribution.

Example 8-8 Crankshaft Bearings

In a random sample of 85 automobile engine crankshaft bearings, 10 have a surface finish that is rougher than the specifications allow. Construct a 95% two-sided confidence interval for *p.*

A point estimate of the proportion of bearings in the population that exceeds the roughness specification is $\hat{p} = x/n = 10/85 = 0.12$

A 95% two-sided confidence interval for *p* is computed from Equation 8-11 as

$$\hat{p} - z_{0.025}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p} + z_{0.025}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$0.12 - 1.96\sqrt{\frac{0.12(0.88)}{85}} \le p \le 0.12 + 1.96\sqrt{\frac{0.12(0.88)}{85}}$$

$$0.0509 \le p \le 0.2243$$

Interpretation: This is a wide CI. Although the sample size does not appear to be small (*n* = 85), the value of is fairly small, which leads to a large standard error for contributing to the wide CI.

Choice of Sample Size

Sample size for a specified error on a binomial proportion :

If we set $E = z_{\alpha/2}\sqrt{p(1-p)/n}$ *and* solve for *n,* the appropriate sample size is

$$n = \left(\frac{z_{\alpha/2}}{E}\right)^2 p(1-p) \qquad (8\text{-}12)$$

The sample size from Equation 8-12 will always be a maximum for *p = 0.5* [that is, *p(1 − p) ≤ 0.25* with equality for *p = 0.5*], and can be used to obtain an upper bound on *n.*

$$n = \left(\frac{z_{\alpha/2}}{E}\right)^2 (0.25) \qquad (8\text{-}13)$$

Example 8-9 Crankshaft Bearings

Consider the situation in Example 8-8. How large a sample is required if we want to be 95% confident that the error in using $\hat{p}$ to estimate *p* is less than 0.05?

Using $\hat{p} = 0.12$ as an initial estimate of *p*, we find from Equation 8-12 that the required sample size is

$$n = \left(\frac{z_{0.025}}{E}\right)^2 \hat{p}(1-\hat{p}) = \left(\frac{1.96}{0.05}\right)^2 0.12(0.88) \cong 163$$

If we wanted to be *at least* 95% confident that our estimate $\hat{p}$ of the true proportion *p* was within 0.05 regardless of the value of *p*, we would use Equation 8-13 to find the sample size

$$n = \left(\frac{z_{0.025}}{E}\right)^2 (0.25) = \left(\frac{1.96}{0.05}\right)^2 (0.25) \cong 385$$

Interpretation: If we have information concerning the value of *p*, either from a preliminary sample or from past experience, we could use a smaller sample while maintaining both the desired precision of estimation and the level of confidence.

Approximate One-Sided Confidence Bounds on a Binomial Proportion

The approximate 100(1 − α)% lower and upper confidence bounds are

$$\hat{p} - z_{\alpha}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \qquad \text{and} \qquad p \le \hat{p} + z_{\alpha}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \qquad (8\text{-}14)$$

respectively.

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Example 8-10 The Agresti-Coull CI on a Proportion

Reconsider the crankshaft bearing data introduced in Example 8-8. In that example we reported that $\hat{p} = 0.12$ and $n = 85$. The 95% CI was $0.0509 \le p \le 0.2243$. Construct the new Agresti-Coull CI.

$$UCL = \frac{\hat{p} + \frac{z_{\alpha/2}^2}{2n} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n} + \frac{z_{\alpha/2}^2}{4n^2}}}{1 + z_{\alpha/2}^2 / n}$$

$$= \frac{0.12 + \frac{1.96^2}{2(85)} + 1.96\sqrt{\frac{0.12(0.88)}{85} + \frac{1.96^2}{4(85)^2}}}{1 + (1.96^2/85)}$$

$$= 0.2024$$

$$LCL = \frac{\hat{p} + \frac{z_{\alpha/2}^2}{2n} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n} + \frac{z_{\alpha/2}^2}{4n^2}}}{1 + z_{\alpha/2}^2 / n}$$

$$= \frac{0.12 + \frac{1.96^2}{2(85)} - 1.96\sqrt{\frac{0.12(0.88)}{85} + \frac{1.96^2}{4(85)^2}}}{1 + (1.96^2/85)}$$

$$= 0.0654$$

Interpretation: The two CIs would agree more closely if the sample size were larger.

8-5 Guidelines for Constructing Confidence Intervals

Table 8-1 provides a simple road map for appropriate calculation of a confidence interval.

TABLE • 8-1 The Roadmap for Constructing Confidence Intervals One-Sample Case

Parameter to Be Bounded by the Confidence Interval or Tested with a Hypothesis?	Symbol	Other Parameters?	Confidence Interval Section
Mean of normal distribution	μ	Standard deviation σ known	8-1
Mean of arbitrary distribution with large sample size	μ	Sample size large enough that central limit theorem applies and σ is essentially known	8-1.5
Mean of normal distribution	μ	Standard deviation σ unknown and estimated	8-2
Variance (or standard deviation) of normal distribution	σ^2	Mean μ unknown and estimated	8-3
Population proportion	p	None	8-4

8-6 Tolerance and Prediction Intervals

8-6.1 Prediction Interval for Future Observation

A 100 $(1 - \alpha)$% prediction interval (PI) on a single future observation from a normal distribution is given by

$$\bar{x} - t_{\alpha/2,n-1}\, s\sqrt{1+\frac{1}{n}} \le X_{n+1} \le \bar{x} + t_{\alpha/2,n-1}\, s\sqrt{1+\frac{1}{n}} \qquad (8\text{-}15)$$

The prediction interval for X_{n+1} will always be longer than the confidence interval for μ.

Example 8-11 Alloy Adhesion

The load at failure for n = 22 specimens was observed, and found that $\bar{x} = 13.71$ and s = 3.55. The 95% confidence interval on μ was $12.14 \le \mu \le 15.28$. Plan to test a 23rd specimen.

A 95% prediction interval on the load at failure for this specimen is

$$\bar{x} - t_{\alpha/2,n-1} s\sqrt{1+\frac{1}{n}} \le X_{n+1} \le \bar{x} + t_{\alpha/2,n-1} s\sqrt{1+\frac{1}{n}}$$

$$13.71 - (2.080)3.55\sqrt{1+\frac{1}{22}} \le X_{23} \le 13.71 + (2.080)3.55\sqrt{1+\frac{1}{22}}$$

$$6.16 \le X_{23} \le 21.26$$

Interpretation: The prediction interval is considerably longer than the CI. This is because the CI is an estimate of a parameter, but the PI is an interval estimate of a single future observation.

8-6.2 Tolerance Interval for a Normal Distribution

A **tolerance interval** for capturing at least γ% of the values in a normal distribution with confidence level $100(1 - \alpha)\%$ is

$$\bar{x} - ks, \qquad \bar{x} + ks$$

where *k* is a tolerance interval factor found in Appendix Table XII. Values are given for $\gamma = 90\%$, 95%, and 99% and for 90%, 95%, and 99% confidence.

Example 8-12 Alloy Adhesion

The load at failure for n = 22 specimens was observed, and found that $\bar{x} = 13.71$ and s = 3.55. Find a tolerance interval for the load at failure that includes 90% of the values in the population with 95% confidence.

From Appendix Table XII, the tolerance factor k for n = 22, γ = 0.90, and 95% confidence is k = 2.264.

The desired tolerance interval is

$$(\bar{x} - ks \ , \ \bar{x} + ks)$$

$$[13.71 - (2.264)3.55 \ , \ 13.71 + (2.264)3.55]$$

$$(5.67 \ , 21.74)$$

<u>Interpretation:</u> We can be 95% confident that at least 90% of the values of load at failure for this particular alloy lie between 5.67 and 21.74.

Important Terms & Concepts of Chapter 8

Chi-squared distribution

Confidence coefficient

Confidence interval

Confidence interval for a:

- Population proportion
- Mean of a normal distribution
- Variance of a normal distribution

Confidence level

Error in estimation

Large sample confidence interval

1-sided confidence bounds

Precision of parameter estimation

Prediction interval

Tolerance interval

2-sided confidence interval

t distribution